Effect of random forcing in the rheology of granular simple flows

F. Vega Reyes

Departamento de Física, Universidad de Extremadura, 06071 Badajoz, Spain

Abstract. We present in this work the evidence of a new type of steady flows in volume-heated granular gases at low densities. The granular gas is heated in its volume by the action of a stochastic thermostat (white noise). The flow of interest belongs to a class of flows that we call 'LTu' class. This class was also observed recently in granular gases heated and sheared from the boundaries. We prove now that in volume-heated granular sheared gases, LTu flows at large inelasticities do also exist and that we can find, for the first time, inelastic Fourier flows belonging to the class. Furthermore, we show that LTu flows in volume-heated granular gases can be made to fall into the Navier-Stokes regime, since non-linear effects can be annihilated if noise intensity is large enough. This means we can find for the first time LTu Newtonian flows, including USF, independently of the value of inelasticity in the collisions. Our work is based on theoretical considerations and Monte Carlo simulations of the Boltzmann equation (DSMC) for the granular gas.

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INTRODUCTION

Contrary to what happens in a traditional gas, the uniform shear flow (USF) in a granular gas is stationary [1, 2]. Because of its simplicity (it has a linear flow velocity and constant temperature and density) this type of flow has been extensively studied and is used as a reference for studying more complex problems in granular flow. Furthermore, recent results have shown that there is a connection between the USF and the classic Fourier flow in an elastic gas, through a new class of granular flows (called 'LTu') with uniform heat flux [1] (as it happens in the USF and the Fourier flow). However, due to inelastic cooling, the LTu and the USF (which is a special case of LTu flow) are inherently non-Newtonian, and a Navier-Stokes approach is in general not useful [2].

On the other hand, we need some kind of forcing in order to sustain the dynamics of a granular system. But, if the energy is exclusively input from the boundaries, large gradients are in general needed in order to obtain the flow of interest [2]. In this respect, homogeneous random forcing (a stochastic thermostat that injects energy directly to the individual particles [3, 4]) is sometimes useful for the study of granular dynamics without incorporating large gradients due to boundary energy sources, while still observing the phenomena of interest. For example, in thin and dense granular layers the suppression of ordered phases with increasing inelasticity in a vibrated layer is still observed if a random forcing is used instead to heat the system [4]. Nevertheless, the phases in systems randomly heated have new properties due to differences in the energy input [4]. Also, it is a question of interest if we still can find LTu flows when a random volume forcing is applied to the granular gas.

We present in this work results from the Direct Simulation Monte Carlo (DSMC) method of the Boltzmann and Enskog equations of the granular gas (low and moderate densities, respectively), with an additional (Gaussian white-noise) random forcing term [4]. The objective of our work is twofold: first, to show that the LTu class, recently reported for a granular gas heated from the boundaries [2], exists also in a granular system with an additional random volume forcing term; second, to study the hydrodynamic properties of the LTu flow, and of the USF as a particular case, as a function of noise intensity. We will show that non-Newtonian effects characteristic of these steady states, such as normal stress differences, can be concealed if the noise intensity is sufficiently large, even for large values of the inelasticity (see Figure 1 for results in the USF). Our results will allow us to study flows like the USF and (more generally) the LTu class in the context of the Navier-Stokes (NS) equations [5] for the whole range of values of inelasticities, whose transport coefficients are well known. Furthermore, also NS stability criteria can be obtained in a straightforward way [6].

We will also show that in volume-heated granular gases we can find a new type of LTu flow, for which the shear rate is zero but the inelasticity is finite. Until now LTu flows with zero shear rate were found only in the elastic limit [2].

THEORY

We study in this work a system of smooth hard spheres of diameter σ and mass m, that collide inelastically. If a pair ij collides, postcollisional velocities of this pair are given by the expressions: $\mathbf{v}_{i,j} = \mathbf{v}''_{i,j} \mp \frac{1}{2}(1+\alpha)(\mathbf{v}''_{ij}\cdot\hat{\sigma}_{ij})\hat{\sigma}_{ij}$, where $\mathbf{v}''_{i,j}$ are the precollisional velocities of particles i and j respectively and $\mathbf{v}_{i,j}$ their postcollisional velocities, $\mathbf{v}''_{ij} = \mathbf{v}''_i - \mathbf{v}''_j$ and $\hat{\sigma}_{ij}$ is a three-dimensional unit vector in the line that joins, in the collision, both sphere centers (from i to j). Also, $\alpha \in [0,1]$ is the coefficient of normal restitution, being $\alpha = 1$ the elastic collision limit. The corresponding kinetic equation for a granular gas, up to moderate densities, has the following form (Enskog kinetic equation, plus a random forcing term):

$$\frac{\partial}{\partial t} f(\mathbf{r}_i,\mathbf{v}_i,t) + \nabla\cdot\mathbf{v}_i f(\mathbf{r}_i,\mathbf{v}_i,t) + \mathscr{F} f(\mathbf{r}_i,\mathbf{v}_i,t) = \chi I[\mathbf{r}_i,\mathbf{v}_i|f(t),f(t)], \tag{1}$$

where

$$I[\mathbf{r}_i,\mathbf{v}_i|f(t),f(t)] \equiv \sigma^{d-1}\int d\mathbf{v}_j \int d\hat{\sigma} \qquad \Theta(\mathbf{v}_{ij}\cdot\hat{\sigma})(\mathbf{v}_{ij}\cdot\hat{\sigma}_{ij}) \times \\ \left[\alpha^{-2} f(\mathbf{v}''_i,\mathbf{r}_i,t) f(\mathbf{r}_i-\sigma_{ij},\mathbf{v}''_j,t) - f(\mathbf{v}_i,t) f(\mathbf{r}_i+\sigma_{ij},\mathbf{v}_j,t)\right], \tag{2}$$

with $\sigma_{ij} = \sigma\hat{\sigma}_{ij}$, and where χ is the pair correlation function at contact [7], that in the low density limit tends to one.

The operator $\mathscr{F}$ represents the action of an external volume force. We chose the following form for $\mathscr{F}$ [3]

$$\mathscr{F} f(\mathbf{r}_i,\mathbf{v}_i,t) = -\frac{\xi_0^2}{2}\left(\frac{\partial}{\partial\mathbf{v}_i}\right)^2 f(\mathbf{r}_i,\mathbf{v}_i,t), \tag{3}$$

where ξ_0 is the noise intensity. Operator $\mathscr{F}$ corresponds to a stochastic force $\mathbf{F}_i$ on particle i with the form of a Gaussian white noise [3]

$$\langle \mathbf{F}_i(t)\rangle = \mathbf{0}, \quad \left\langle \mathbf{F}_i(t)\mathbf{F}_j(t')\right\rangle = \mathrm{I} m^2 \xi_0^2 \delta_{ij}\delta(t-t'), \tag{4}$$

where I is the 3D identity matrix.

In addition, we suppose our system is enclosed between two infinite parallel walls (separated by a distance h) from which we shear and heat the granular gas: both walls may have relative velocities ΔU and also different temperatures (temperature difference ΔT between both walls). In this type of configuration, stationary solutions only depend on one space coordinate -the perpendicular direction to both walls (say, y)-. We define as units for length and time the mean free path $\lambda_r = \lambda'(\sqrt{2} n_r \sigma^2)^{-1}$, and the collision frequency $\nu_r = \sqrt{2T_r/m} n_r \sigma^2/\lambda'$, where $\lambda' = \sqrt{2}\frac{5}{16}\pi^{-1/2}$ and n_r and T_r are the density and temperature of the gas next to the colder wall. We set the y coordinate of the colder wall at $y = -h/2$. In what follows, quantities are nondimensionalized by the choice of units m (particle mass), λ_r and ν_r^{-1} for mass, length and time respectively.

In our system, and for a low density granular gas, the equation of energy balance reads [5]

$$-\frac{\partial q_y}{\partial y} = \frac{3}{2}(\zeta - \Lambda) - \eta^*(\alpha) a^2, \tag{5}$$

where the term $\Lambda > 0$ comes from the random energy source term [5] and $a = -P_{xy}/\eta^*(\alpha) = \nu^{-1}\partial_y u_x$ is the shear rate, being $\eta^*(\alpha) = \eta(\alpha)/\eta_{NS}$ where subscript NS indicates Navier-Stokes (NS) transport coefficient of the elastic gas. P_{xy} is the xy component of the stress tensor. For steady solutions in our system geometry, a is constant, and also the hydrostatic pressure p, (and thus, in our reduced magnitudes $p = 1$) [6].

By definition, the LTu class of flows occurs when the heat flux in (5) is constant ($\partial_y q_y = 0$) [2]. The existence of the LTu class in boundary-heated granular gases ($\Lambda = 0$) was first predicted, at Navier-Stokes order (i.e., the quasielastic limit), in a recent work [6] and has also been found for finite inelasticities in a subsequent work [2]. For $\Lambda = 0$, condition $\partial_y q_y = 0$ is possible only if there is an equilibration in (5) of the inelastic cooling term $3\zeta T/2$ (energy sink) with the viscous heating term $-\eta^*(\alpha)a^2$. It is straightforward to prove this occurs for a specific value of a that depends only on α [6]. It is also straightforward that when $\partial_y q_y = 0$ we obtain also [6]

$$\frac{1}{\nu}\frac{\partial T}{\partial y} = A = \text{const}, \tag{6}$$

where $A = (T(L/2) - T(-L/2))/L$ (with $L = h\nu$) is a constant parameter controlled by the boundary conditions (specifically thorough ΔT) [6]. It is clear from (6) that for the subclass $A = 0$ we get a constant temperature (and density) profile; i.e. the rather well known uniform shear flow (USF) of granular gases is a special case of LTu flow [2].

The pending question is if we can find the same type of LTu steady solutions in a volume-heated granular gas ($\Lambda \neq 0$). Our results from Monte Carlo simulation of the kinetic equation (1) of the granular gas will confirm we can find LTu flows in a volume-heated granular gas, at least with a random forcing term of the form (3). We will pay special attention to the USF case, showing that it has peculiar properties respect to the $\Lambda = 0$ case. To begin with, the shear rate point for which this occurs would be logically different. It will actually have a tendency to be smaller since the random forcing compensates partly for the collisional cooling (see (5)). As we will see, if the uniform heating term is too strong, it would itself compensate for the collisional cooling, causing the viscous heating to exceed condition $\partial_y q_y = 0$ even in the limit $a \to 0$. Thus, we would expect that beyond a threshold value of noise intensity, no steady solution for a shear flow will be possible.

SIMULATION

We obtain in this work a series of numerical solutions to the Boltzmann equation by means of the Direct Simulation Monte Carlo method, which was devised by Bird [8] and later applied to the Boltzmann equation for granular gases [3]. The method was also extended to the Enskog equation for moderate density gases [9]. We briefly describe in this section the basics of the algorithm. The simulation advances in small time steps dt (small compared to the inverse collision frequency ν_r^{-1}), which are iterated until a stationary solution is reached. The algorithm is composed by two basic procedures at each time step. The first one is free streaming, during which all hard spheres are displaced in straight lines with constant velocities. Collisions between particles occur only in the next procedure, particle collisions, during which a sample of $N\omega_{max}dt/2 = (4\pi\chi\sigma^2 n)|\mathbf{v}_{ij}\cdot\hat{\sigma}_{ij}|_{max}$ pairs is chosen at random without preference, being $\omega_{max} \equiv (4\pi\chi\sigma^2 n)|\mathbf{v}_{ij}\cdot\hat{\sigma}_{ij}|_{max}$ is an upper bound estimate of the particle collision probability per unit time $\omega_{ij} \equiv (4\pi\chi\sigma^2 n)|\mathbf{v}_{ij}\cdot\hat{\sigma}_{ij}|$, being $\mathbf{v}_{ij} = \mathbf{v}_i - \mathbf{v}_j$. The upper bound probability ω_{max} is proportional to the magnitude $|\mathbf{v}_{ij}\cdot\hat{\sigma}_{ij}|_{max}$, for which a value of $|\mathbf{v}_{ij}\cdot\hat{\sigma}_{ij}|_{max} \sim 8$ is enough in most simulations. In any case, if a value $|\mathbf{v}_{ij}\cdot\hat{\sigma}_{ij}| > |\mathbf{v}_{ij}\cdot\hat{\sigma}_{ij}|_{max}$ is found in a time step, where $\hat{\sigma}_{ij}$ is picked from a uniform distribution, then this upper bound is updated to the new maximum. A potential collision pair ij is accepted only if $\omega_{ij}/\omega_{max} > p_r$, where $p_r \in (0,1)$ is a random number generated from a uniform distribution. In closed systems like ours, a third procedure dealing with boundary conditions is also needed. Also, there is an additional procedure due to the random forcing term, that acts by adding a velocity $\mathbf{w}_i$ to each particle velocity $\mathbf{v}_i$ being $\mathbf{w}_i$ randomly drawn from the Gaussian probability distribution [3]

$$P(\mathbf{w}) = (2\pi\xi_0^2 dt)^{3/2} e^{-w^2/2\xi_0^2 dt}. \tag{7}$$

For the method to be meaningful, the simulation time step dt must be small compared to the inverse collision frequency ν_r^{-1} and the collision sample must be chosen among particles belonging to a cell whose characteristic length is small compared to the mean free path. The cell needs to have a finite size only in the space variables in those space directions on which the stationary solution of the problem depends. For example, in our problem, the stationary solutions depend only on the y space variable and thus our cells in the simulation are 1-dimensional. Furthermore, for the special case of the USF (linear profile $u_x(y) = ay$), the problem is space-independent in the local Lagrangian frame (i.e., if the distribution function is expressed in terms of the peculiar velocities) and for this reason we do not need to split the system into simulation cells [9]. Thus, we used a cell size $dy = 0.02\ \lambda_r$ for generic LTu flows ($\Delta T \neq 0$) and just one cell for USF simulations ($\Delta T = 0$) and a time step $dt = 0.003\ \nu_r^{-1}$ for all simulations. Apart from velocities (and positions in non-homogeneous simulations), only temperature (and density, idem) needs to be measured every several time steps. The rest of magnitudes (stress tensor, heat flux, etc.) are measured only in longer time steps.

As we said, since for the USF n, T are constant and $u_x(y)$ is linear, Monte Carlo simulations of this state can be performed as space-independent if referred to the Langrangian frame (i.e., in the Monte Carlo algorithm, peculiar velocities $\mathbf{V}_i = \mathbf{v}_i - \mathbf{u}$, being $\mathbf{u}$ the flow velocity, are simulated instead) [9]. This obviously makes simulations of the USF much faster than those of generic LTu flows ($A \neq 0$), for which the system is not homogeneous in the Lagrangian frame. We have currently obtained USF steady states in DSMC simulations runs just 20 s long [10]. Furthermore, the choice of this type of fast simulations is very convenient even for accessing $A \neq 0$ LTu flows since the rheology of both $A = 0$ and $A \neq 0$ LTu flows is the same (only the aspect of $\mathbf{u}_x(y)$ and $T(y)$ distinguishes them [6, 2]). For the

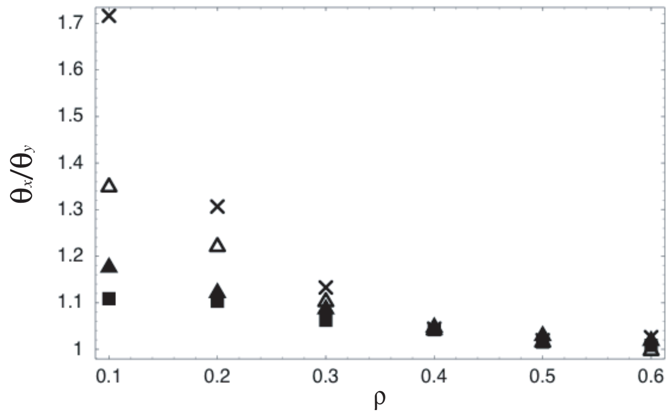

FIGURE 1. Normal stress ratio θ_x/θ_y as a function of solid fraction ρ, for increasing noise intensity ($\xi_0^2 = 5, 2.5, 1, 0$), from top to bottom) for a granular gas of inelastic hard spheres with a coefficient of normal restitution $\alpha = 0.6$. As it can be seen, the ratio tends to 1 as the noise intensity ξ_0 increases, thus showing that the random forcing conceals non-linear effects. The effect is more prominent for low density gases.

more general case $A \neq 0$, simulations are not as straightforward. First of all, it is necessary to subdivide the system in simulation cells, as explained above, and secondly, we need to look for the exact LTu point by properly adjusting the wall relative velocities for a given value of α, and also the wall temperature difference ΔT. Usually, a rather large number of simulations for a given value of α are performed before the LTu condition is found because the relations $a(\Delta U)$ (velocity slip) and $A(\Delta T)$ (wall temperature jump) are not known a priori; i.e., we cannot directly fix a nor A. We will use the expression $T(h/2) - T(-h/2)$ to denote the temperature difference of the granular fluid next to both walls, which is different to the wall temperature difference itself ΔT.

DISCUSSION AND CONCLUSION

We present in Figure 1 results for a simulation of the steady USF for a granular gas with $\alpha = 0.6$. As we can see, the normal stress differences tend to decrease for increasing ξ_0, much in the line of the discussion in the Introduction section. Therefore, this is clearly an indication of an approach to Newtonian fluid-like behaviour as the effects of random forcing become more important, even for a large inelasticity ($\alpha = 0.6$). As we can see, this effect is much stronger in the range of low densities. This is no limitation since we are interested specifically in this limit for the study of the properties of LTu flows. The same effect of non-linear behaviour reduction with increasing ξ_0 is obtained in $A \neq 0$ simulations.

Figure 2 presents hydrodynamic profiles showing clearly that in effect LTu flows still can be found in a granular gas when a random forcing term $\mathscr{F}$ is added. More concretely, Figure 2 (a) shows the characteristic linear $T(u_x)$ profiles of the LTu flows [6, 2] and Figure 2 (b) represents heat flux profiles, for which as we can see, both components are clearly constant. In Figure 3 we show, for increasing ξ_0, the threshold shear rate a, reduced viscosity $\eta^*(\alpha) = \eta/\eta_{NS}$ and the ratio θ_x/θ_y of the reduced normal stress tensor components $\theta_i = P_{ii}/p$ for a granular gas with $\alpha = 0.8$. It is interesting to note: first, that normal stress differences tend to vanish for sufficiently high ξ_0, and second, that $a \to 0$ for a certain critical value of ξ_0. We actually found that beyond this critical value is not possible to find flows with the properties of the LTu class. We suggest this is due to the aforementioned 'overheating' effect of the random forcing term for larger ξ_0, not allowing for the existence of a finite threshold shear rate a that compensates the inelastic cooling. This apparently annoying effect for finding LTu flows has nevertheless a very interesting collateral consequence: we can find constant heat flux steady states in Fourier flow-like configurations (because $a = 0$). In effect, for instance the last point in the shear rate a curve in Figure 2 is already very close to this situation, which we found to occur, for $\Delta T = 5$ and $\alpha = 0.8$, at $\xi_0^2 = 4.25$. This new state should not be mistaken with the steady state of a volume-heated granular gas [3]: our system is now not homogeneous and temperature profiles have the characteristic curvature of Fourier-like flows, as it happens for regular LTu flows. What we have found is a new element of the LTu flow class with the distinctive property of having zero shear rate but contrary to the elements of this type in a previous work [2], they occur for inelastic gases instead of elastic gases.

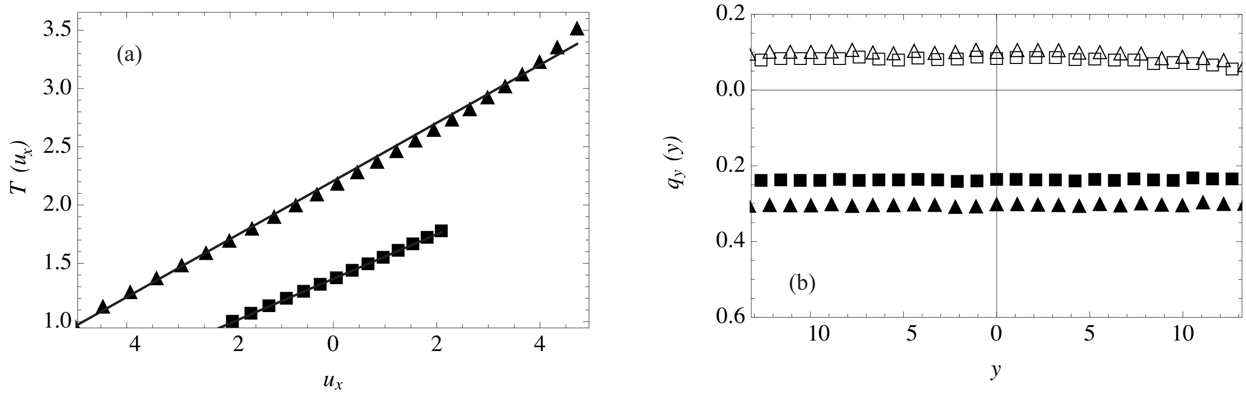

FIGURE 2. Characteristic profiles of LTu flows in a volume-heated granular gas, for $\alpha = 0.8$, and two different simulation sets: $\xi_0^2 = 0.5$, $T(h/2) - T(-h/2) = 3.5$ (triangles) and $\xi_0^2 = 2.5$, $T(h/2) - T(-h/2) = 2.0$ (squares)(a) Linear $T(u_x)$ profiles, characteristic of the LTu flow class, as extracted from DSMC data, with $\rho \to 0$. For both series, $h = 15$. (b) Constant heat flux profiles, for both components: q_x (open symbols), and q_y (solid symbols).

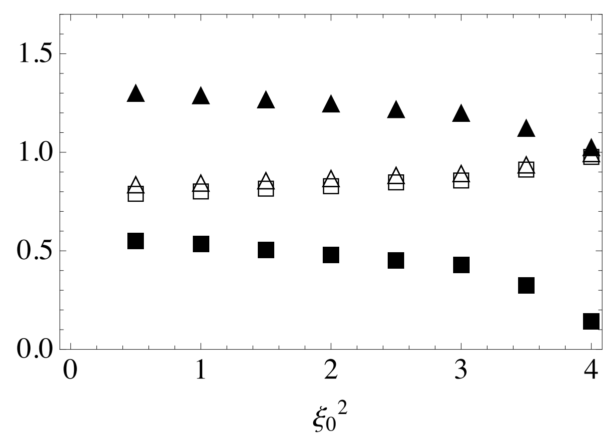

FIGURE 3. Plot of $a(\alpha)$ (solid squares), $\eta^*(\alpha)$ (open squares), $\theta_x(\alpha)$ (solid triangles), $\theta_y(\alpha)$ (open triangles) in LTu flows for a volume-heated granular gas with $\alpha = 0.8$, $h = 15$, and $T(h/2) - T(-h/2) = 3.5$, as a function of noise intensity, as obtained from Monte Carlo simulations, with $\rho \to 0$. Note that for increasing higher ξ_0 the normal stress differences (a non-linear effect) tend to be concealed. As we can see, the shear rate needs to be higher for smaller noise intensity since in this case more stronger compensation of collisional cooling is required to viscous heating.

In Figure 4 we present results for the heat flux transport coefficients, also as a function of ξ_0. Both coefficients are reduced with the NS thermal conductivity of a gas of elastic hard spheres; i.e. $\lambda^*(\alpha) = \lambda(\alpha)/\lambda_{NS}$ and $\phi^*(\alpha) = \phi(\alpha)/\lambda_{NS}$, with $\lambda(\alpha) = -\partial_y T/q_y$, $\phi(\alpha) = \partial_y T/q_x$ (the cross coefficient $\phi \neq 0$ is a non-linear effect [2]). As it happens for the previous figure, it is quite clear that non-linear behaviour tends to decrease as we increase ξ_0. In effect, the cross thermal conductivity coefficient $\phi^*(\alpha)$ tends to vanish as we approach the critical value of ξ_0 for which the new LTu element described above occurs, whereas the thermal conductivity $\lambda^*(\alpha) \to 1$ (λ approches the behaviour of the elastic gas).

To summarize, following recent results from previous theoretical works [6, 2], we have studied a granular gas system in the Couette flow geometry. As in these previous works, the granular gas is heated and sheared from the boundaries, but we also added a random volume forcing term. As we have seen, we still can find LTu flows (i.e., flows with constant heat flux) in this type of configuration. In effect, we found steady states for which, in addition to the general properties of Couette gas flows (constant shear rate a and hydrostatic pressure p) we found the distinctive properties of LTu flows: $T(u_x)$ is linear and $q_{x,y}(y)$ are constant. Also, and like in previous work [2] we found that in general these LTu flows have highly non-linear behavior. But, contrary to results in previous work [2], non-linear effects can be concealed since we can also find newtonian-like fluid behaviour, even for large inelasticities, if we increase the

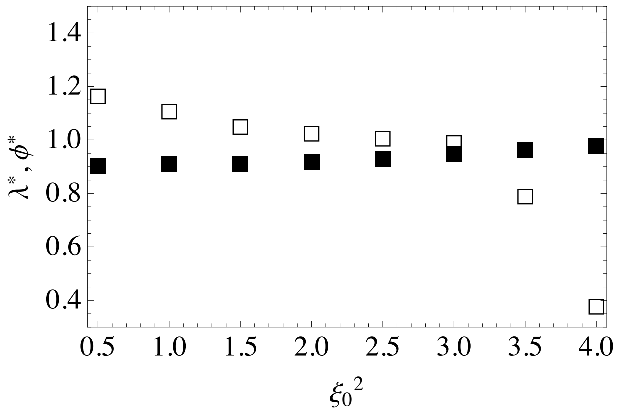

FIGURE 4. Plot of heat flux coefficients λ^* (solid symbols) and ϕ^* (open symbols) for LTu flows as extracted from DSMC data ($\rho \to 0$), as a function of noise intensity ξ_0.

intensity of the random forcing term. Furthermore for a critical value of this noise intensity, we find a new element of the LTu flow class not detected previously. It consists of a Fourier-like flow (i.e., no shear) but, contrary to what was found in previous works, it occurs in inelastic gases (Fourier flow belonged to the LTu class in previous work only in the elastic limit [2]). Therefore, we have resolved that LTu Fourier flows are not necessarily in the elastic limit but can occur at finite inelasticities, if the gas is heated in the volume by means of a random forcing term and for specific intensity values of this force.

More interestingly, since the behaviour of the hydrodynamics of the LTu flows in a volume-heated granular gas approaches the Newtonian fluid behaviour, previous results on Navier-Stokes hydrodynamics in volume-heated granular gases [5] could be used to describe theoretically these new LTu flows, (and also non-LTu flows if noise intensity is large but does not coincide with the critical value for which heat flux is constant). We expect in forthcoming work to study further these questions.

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